

# TOPOLOGICAL PROPERTIES OF TAIMANOV SEMIGROUPS

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**ABSTRACT.** A semigroup  $T$  is called *Taimanov* if  $T$  contains two distinct elements  $0, \infty$  such that  $xy = \infty$  for any distinct points  $x, y \in T \setminus \{0, \infty\}$  and  $xy = 0$  in all other cases. We prove that any Taimanov semigroup  $T$  has the following topological properties: (i) each  $T_1$ -topology with continuous shifts on  $T$  is discrete; (ii)  $T$  is closed in each  $T_1$ -topological semigroup containing  $T$  as a subsemigroup; (iii) every non-isomorphic homomorphic image  $Z$  of  $T$  is a zero-semigroup and hence  $Z$  is a topological semigroup in any topology on  $Z$ .

We shall follow the terminology of [5, 8, 10, 20].

The problem of non-discrete (Hausdorff) topologization of infinite groups was posed by Markov [17]. This problem was resolved by Ol'shanskiy [19] who constructed an infinite countable group  $G$  admitting no non-discrete Hausdorff group topologies. On the other hand, Zelenyuk [23] proved that each group  $G$  admits a non-discrete shift-continuous Hausdorff topology  $\tau$  with continuous inversion  $G \rightarrow G, x \mapsto x^{-1}$ . In [1, 2.10] it was observed that Ol'shanskiy construction can be modified to produce for every non-zero  $m \in \mathbb{Z} \setminus \{-2^n, 2^n : n \in \omega\}$  a countable infinite group  $G_m$  admitting no non-discrete shift-continuous topology with continuous  $m$ -th power map  $G_m \rightarrow G_m, x \mapsto x^m$ .

Studying the topologizability problem in the class of inverse semigroups, Eberhart and Selden [9] proved that every Hausdorff semigroup topology on the bicyclic semigroup  $\mathcal{C}(p, q)$  is discrete. This result was generalized by Bertman and West [4] who proved that every Hausdorff shift-continuous topology on  $\mathcal{C}(p, q)$  is discrete. In [2, 3, 6, 7, 11, 12, 13, 14, 15, 16, 18] these topologizability results were extended to some generalizations of the bicyclic semigroup.

Studying the topologizability problem in the class of commutative semigroups [22], Taimanov in [21] constructed a commutative semigroup  $\mathfrak{A}_\kappa$  of arbitrarily large cardinality  $\kappa$ , which admits no non-discrete Hausdorff semigroup topology, but any non-isomorphic homomorphic image  $Z$  of  $T$  is a zero-semigroup and hence is a topological semigroup in any topology on  $Z$ . We recall that a semigroup  $Z$  is a *zero-semigroup* if the set  $SS = \{xy : x, y \in X\}$  is a singleton  $\{z\}$ . In this case the element  $z$  is the *zero-element* of the semigroup  $S$ , i.e., a (unique) element  $z \in S$  such that  $xz = z = zx$  for all  $x \in S$ . In this paper we improve the mentioned Taimanov's result proving that the Taimanov semigroup  $\mathfrak{A}_\kappa$  admits no non-discrete shift-continuous  $T_1$ -topologies and is closed in any  $T_1$ -topological semigroup containing  $\mathfrak{A}_\kappa$  as a subsemigroup. First we give an abstract definition of a Taimanov semigroup.

**Definition 1.** A semigroup  $T$  is called *Taimanov* if it contains two distinct elements  $0_T, \infty_T$  such that for any  $x, y \in T$

$$x \cdot y = \begin{cases} \infty_T & \text{if } x \neq y \text{ and } x, y \in T \setminus \{0_T, \infty_T\}; \\ 0_T & \text{if } x = y \text{ or } \{x, y\} \cap \{0_T, \infty_T\} \neq \emptyset. \end{cases}$$

The elements  $0_T, \infty_T$  are uniquely determined by the algebraic structure of  $T$ :  $0_T$  is a (unique) zero-element of  $T$ , and  $\infty_T$  is the unique element of the set  $TT \setminus \{0_T\}$ .

It follows that each Taimanov semigroup  $T$  is commutative. Concrete examples of Taimanov semigroups can be constructed as follows.

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**Example 1.** For any non-zero cardinal  $\kappa$  the set  $\kappa \cup \{\kappa\}$  endowed with the commutative semigroup operation defined by

$$xy = \begin{cases} \kappa & \text{if } x \neq y \text{ and } x, y \in T \setminus \{0, \kappa\}; \\ 0 & \text{if } x = y \text{ or } \{x, y\} \cap \{0, \kappa\} \neq \emptyset. \end{cases}$$

is a Taimanov semigroup of cardinality  $1 + \kappa$ . Here we identify the cardinal  $\kappa$  with the set  $[0, \kappa)$  of ordinals, smaller than  $\kappa$ .

**Proposition 1.** *Two Taimanov semigroups are isomorphic if and only if they have the same cardinality.*

*Proof.* Given two Taimanov semigroups  $T, S$  of the same cardinality, observe that any bijective map  $f : T \rightarrow S$  with  $f(0_T) = 0_S$  and  $f(\infty_T) = \infty_S$  is an algebraic isomorphism of  $T$  onto  $S$ .  $\square$

In this paper we show that any Taimanov semigroup  $T$  has the following topological properties:

- (1) every shift-continuous  $T_1$ -topology on  $T$  is discrete;
- (2)  $T$  is closed in each  $T_1$ -topological semigroup containing  $T$  as a subsemigroup;
- (3) every non-isomorphic homomorphic image  $Z$  of  $T$  is a zero-semigroup and hence any topology on  $Z$  turns it into a topological semigroup.

The first statement generalizes the original result of Taimanov [21] and is proved in the following proposition.

**Proposition 2.** *Every shift-continuous  $T_1$ -topology  $\tau$  on any Taimanov semigroup  $T$  is discrete.*

*Proof.* The statement is trivial if the semigroup  $T$  is finite. So, assume that  $T$  is infinite. The topology  $\tau$  satisfies the separation axiom  $T_1$  and hence contains an open set  $U \subset X$  such that  $0_T \in U$  and  $\infty_T \notin U$ .

First we prove that the points  $0_T$  and  $\infty_T$  are isolated in  $T$ . Chose any point  $x \in T \setminus \{0_T, \infty_T\}$  and observe that  $x \cdot 0_T = x \cdot \infty_T = 0_T \in U$ . By the shift-continuity of the topology  $\tau$ , there exist neighborhoods  $U_0 \in \tau$  of  $0_T$  and  $U_\infty \in \tau$  of  $\infty_T$  such that  $(x \cdot U_0) \cup (x \cdot U_\infty) \subset U$ . We claim that  $U_0 \setminus \{x, \infty_T\} = \{0_T\}$  and  $U_\infty \setminus \{x, 0_T\} = \{\infty_T\}$ . In the opposite case we could find a point  $y \in (U_0 \cup U_\infty) \setminus \{x, 0_T, \infty_T\}$  and conclude that  $\infty_T = xy \in x \cdot (U_0 \cup U_\infty) \subset U \subset T \setminus \{\infty_T\}$ , which is a desired contradiction showing that the points  $0_T$  and  $\infty_T$  are isolated in  $T$ .

To show that each point  $x \in T \setminus \{0_T, \infty_T\}$  is isolated in the topology  $\tau$ , observe that  $xx = 0_T \in T \setminus \{\infty_T\} \in \tau$  and use the shift-continuity of the topology  $\tau$  to find a neighborhood  $U_x \in \tau$  of  $x$  such that  $xU_x \subset T \setminus \{\infty_T\}$ . Assuming that  $U_x \neq \{x\}$  we can choose any point  $y \in U_x \setminus \{x\}$  and conclude that  $\infty_T = xy \in T \setminus \{\infty_T\}$ , which is a contradiction showing that  $U_x = \{x\}$  and hence the point  $x$  is isolated in the topology  $\tau$ .  $\square$

The following example shows that any infinite Taimanov semigroup admits a non-discrete semigroup  $T_0$ -topology.

**Example 2.** For any infinite Taimanov semigroup  $T$  the family of subsets

$$\tau := \{U \subset T : \text{if } 0_T \in U, \text{ then } \infty_T \in U \text{ and } |T \setminus U| < \omega\}$$

is a  $T_0$ -topology turning  $T$  into a topological semigroup.

A semitopological semigroup  $S$  will be called *square-topological* if the map  $S \rightarrow S, x \mapsto x^2$ , is continuous. It is clear that each topological semigroup is square-topological.

**Theorem 1.** *A Taimanov semigroup  $T$  is closed in any square-topological semigroup  $S$  containing  $T$  as a subsemigroup and satisfying the separation axiom  $T_1$ .*

*Proof.* Assuming that  $T$  is not closed in  $S$ , choose any point  $s \in \bar{T} \setminus T$ . We claim that  $sx = \infty_T$  for any  $x \in T \setminus \{0_T, \infty_T\}$ . Assuming that  $sx \neq \infty_T$  and using the shift-continuity of the  $T_1$ -topology of  $S$ , we can find a neighborhood  $U_s \subset S$  of  $s$  such that  $U_s \cdot x \subset S \setminus \{\infty_T\}$ . Since  $s$  is an accumulation point of the

set  $T$  in  $S$ , there exists a point  $y \in U_s \setminus \{x, 0_T, \infty_T\}$ . For this point  $y$  we get  $\infty_T = yx \in U_s x \subset S \setminus \{\infty_T\}$ , which is a contradiction showing that  $sx = \infty_T$  for any  $x \in T \setminus \{0_T, \infty_T\}$ . Next, we show that  $ss = \infty_T$ . Assuming that  $ss \neq \infty_T$ , we can use the shift-continuity of the  $T_1$ -topology of  $S$  to find a neighborhood  $V_s \subset S$  of  $s$  such that  $sV_s \subset S \setminus \{\infty_T\}$ . Since  $s$  is an accumulation point of the set  $T$  in  $S$ , there exists a point  $x \in V_s \cap T \setminus \{0_T, \infty_T\}$ . For this point  $x$ , we get  $\infty_T = sx \in sV_s \subset S \setminus \{\infty_T\}$ , which is a contradiction showing that  $ss = \infty_T$ . By the separation axiom  $T_1$ , the set  $S \setminus \{0_T\}$  is an open neighborhood of  $\infty_T$  in  $S$ . The continuity of the map  $S \rightarrow S, x \mapsto x^2$ , yields a neighborhood  $W_s \subset S$  such that  $x^2 \in S \setminus \{0_T\}$  for any  $x \in W_s$ . Since  $s$  is an accumulation point of the set  $T$  in  $S$ , there exists a point  $x \in W_s \cap T \setminus \{0_T, \infty_T\}$ . For this point  $x$  we get  $0_T = xx \in S \setminus \{0_T\}$ , which is a desired contradiction showing that the set  $T$  is closed in  $S$ .  $\square$

The following example shows that any infinite Taimanov semigroup admits a (non-closed) embedding into a compact Hausdorff semitopological semigroup and also shows that the continuity of the map  $S \rightarrow S, x \mapsto x^2$ , in Theorem 1 is essential and cannot be replaced by the continuity of the map  $S \rightarrow S, x \mapsto x^m$ , for some  $m \geq 3$ .

**Example 3.** Let  $T$  be a Taimanov semigroup and  $X$  be any  $T_1$ -topological space containing  $T$  as a non-closed dense discrete subspace. Extend the semigroup operation of  $T$  to a binary operation of  $X$  defined by the formula:

$$xy = \begin{cases} 0_T & \text{if } x = y \in T \text{ or } \{x, y\} \cap \{0_T, \infty_T\} \neq \emptyset; \\ \infty_T & \text{otherwise.} \end{cases}$$

Since  $(xy)z = 0_T = x(yz)$  for any  $x, y, z \in X$  the extended operation is associative and turns  $X$  into a commutative semigroup containing  $T$  as a subsemigroup. Observe that for  $a \in \{0_T, \infty_T\}$  the shift  $l_a = r_a : X \rightarrow X, x \mapsto ax = xa = 0_T$ , is constant and hence continuous. For any  $a \in T \setminus \{0_T, \infty_T\}$  the shift  $l_a = r_a : X \rightarrow X, x \mapsto xa = ax$ , is almost constant in the sense that  $l_a^{-1}(\infty_T) = X \setminus \{a, 0_T, \infty_T\}$  and hence is continuous (as the set  $\{a, 0_T, \infty_T\}$  is closed and open in  $X$ ). For any  $a \in X \setminus T$  the shift  $l_a = r_a : X \rightarrow X, x \mapsto xa = ax$ , is almost constant in the sense that  $l_a^{-1}(\infty_T) = X \setminus \{0_T, \infty_T\}$  and hence is continuous. This shows that  $X$  is a semitopological commutative semigroup containing  $T$  as a non-closed dense subsemigroup. Observe also that for every  $m \geq 3$  the map  $X^m \rightarrow X, (x_1, \dots, x_m) \mapsto x_1 \cdots x_m = 0_T$ , is constant and hence continuous. Then the map  $X \rightarrow X, x \mapsto x^m$ , is continuous as well.

**Example 4.** For any topological zero-semigroup  $Z$  with zero  $0_Z$  and any Taimanov semigroup  $T$  endowed with the discrete topology, any map  $h : T \rightarrow Z$  with  $h(0_T) = h(\infty_T) = 0_Z$  is a continuous semigroup homomorphism. Hence there exist many topological (zero-)semigroups containing continuous homomorphic images of Taimanov semigroups as non-closed subsemigroups.

**Proposition 3.** *Any non-isomorphic homomorphic image  $S$  of a Taimanov semigroup  $T$  is a zero-semigroup.*

*Proof.* Fix a non-injective surjective homomorphism  $h : T \rightarrow S$ . If  $f(0_T) = f(\infty_T)$ , then  $SS = f(T) \cdot f(T) = f(TT) = f(\{0_T, \infty_T\}) = \{f(0_T)\}$ , which means that  $S$  is a zero-semigroup. So, assume that  $f(0_T) \neq f(\infty_T)$ . Since  $f$  is not injective, there exist two distinct points  $a, b \in T$  with  $f(a) = f(b)$ . Since  $f(0_T) \neq f(\infty_T)$ , one of the points  $a, b$ , say  $a$ , belongs to  $T \setminus \{0_T, \infty_T\}$ . If  $b \notin \{0_T, \infty_T\}$ , then  $ab = \infty_T$  and  $aa = 0_T$  and hence  $f(\infty_T) = f(ab) = f(a)f(b) = f(a)f(a) = f(aa) = f(0_T)$ , which contradicts our assumption. This contradiction shows that  $b \in \{0_T, \infty_T\}$  and hence  $bc = 0_T$  for any  $c \in T$ .

If  $|T| \geq 4$ , then we can find a point  $c \in T \setminus \{a, 0_T, \infty_T\}$  and conclude that  $f(\infty_T) = f(ac) = f(a)f(c) = f(b)f(c) = f(bc) = f(0_T)$ , which contradicts our assumption. So,  $|T| \leq 3$  and hence  $T = \{a, 0_T, \infty_T\}$  and

$$S = f(T) = \{f(a), f(0_T), f(\infty_T)\} = \{f(b), \{f(0_T), f(\infty_T)\}\} = \{f(0_T), f(\infty_T)\}.$$

Then  $SS = f(\{xy : x, y \in \{0_T, \infty_T\}\}) = \{f(0_T)\}$ , which means that  $S$  is a zero-semigroup.  $\square$

Since the semigroup operation  $Z \times Z \rightarrow \{0_Z\} \subset Z$  of any zero-semigroup  $Z$  is constant and hence is continuous with respect to any topology on  $X$ , Proposition 3 implies the following corollary.

**Corollary 1.** *Every non-isomorphic homomorphic image  $S$  of a Taimanov semigroup is a topological semigroup with respect to any topology on  $S$ .*

We call that a semigroup  $S$  is *algebraically complete* in a class  $\mathcal{S}$  of semitopological semigroups if  $S$  is a closed subsemigroup in each semitopological semigroup  $T \in \mathcal{S}$  containing  $S$  as a subsemigroup. Theorem 1 implies the following

**Corollary 2.** *Each Taimanov semigroup  $T$  is algebraically complete in the class of square-topological semigroups satisfying the separation axiom  $T_1$ . In particular,  $T$  is algebraically complete in the class of  $T_1$ -topological semigroups.*

**Remark 1.** Corollary 1 implies that for any Taimanov semigroup  $T$  and any non-isomorphic surjective homomorphism  $h : T \rightarrow S$  with the infinite image  $S = h(T)$  the semigroup  $S$  is a dense proper subsemigroup of some (compact) Hausdorff topological zero-semigroup.

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